

Linking Hanging Chain Models to Fabrication

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Abstract

The paper traces the development of a digital hanging chain modeler in Java inspired by Antonio Gaudi's physical hanging chain models. More importantly, it demonstrates how fabrication schemas for physical mockups of the digitally simulated hanging chain can be linked to the real time form finding simulation. Fabrication output is an integral part of the iterative process and not a post design process. The current implementation is still limited and currently requires programming for reconfiguration. The paper proposes the link of form finding and fabrication finding and lays out several examples and first steps of how to do so. (Figure 1)

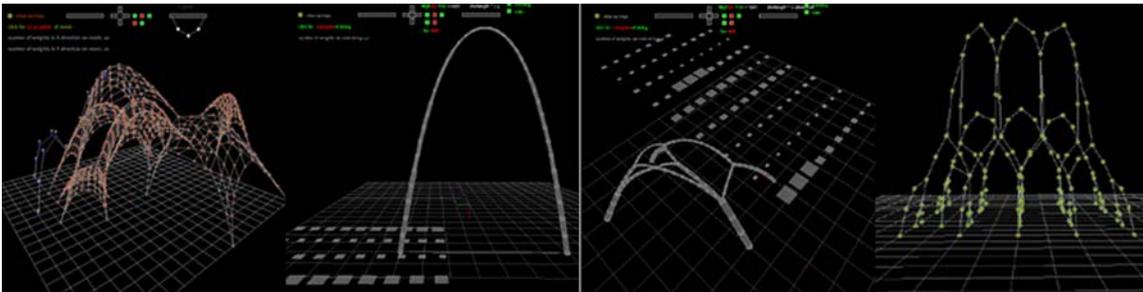


Figure 1. Overview of the tool, showing mesh and string models as well as flattened patterns linked to the live form finding model

1 Introduction

Using physical form finding techniques as an engineering technique has precedents reaching back to the Renaissance. In the early 20th century, Gaudi pioneered them as a design tool in three dimensions using hanging chain models. He produced stunning work of architecture that convince through structural and sculptural elegance. However, little is written about the process whereby the abstract string based chain model with point loads is translated into a volumetric geometry with distributed loads and structures with self-weight. A weak point of Gaudi's technique of the physical hanging chain model is to not give information on the optimal distribution of stress in the material of the built form. (Tomlow et al 1989) The author proposes some techniques to integrate this translation into the design stage of the hanging models.

The paper presents a digital hanging chain model based on a particle spring systems using

Euler and Runge Kutta solvers programmed in Java. The goal is to provide a real-time 3D-modeling environment that allows the design of gravity based forms following the hanging chain principle. By using the same building components it is also possible to model any mesh topology, for instance approximations of Heinz Isler's grid shells or fabric like surfaces. The initial tool was further developed and tested in a workshop setting in order to validate results and provide a more solid implementation and input from students to the tool. The relationship between form finding model and the translation into volumetric form was explored in a series of small models.

2 Motivation

Physical hanging models are very compelling design devices. But a number of factors limit their use for designers. First, they need to be relatively big in scale to give accurate results and to allow measurements with reasonable tolerances. Equilibrium solutions can be scaled if the proportional distribution of mass is kept and the geometry of the lines of forces is scaled proportionally. This holds true even though mass does not scale proportionally to geometric dimensions. So the results of the physical model are usable for a full-scale building design at least as a starting point. But the amount of work needed to construct a fully detailed hanging model is enormous. Also, it is a very time consuming task to adjust the model when larger changes within its geometry occur, as the model is inherently interdependent. A small change can ripple through large parts of the model requiring adjustments which themselves cause shift elsewhere. The model will eventually always find a state of equilibrium, but it is not guaranteed that the new state matches the desired form. This may require several iterative adjustments.

The second major disadvantage is that a physical model is hard to measure accurately and in reasonable time, as measuring requires physically accessing the model. The measurement of forces within the strings of the model is even more difficult, as it requires the installation of strain gauges, which is time consuming and can potentially disturb the model. In addition, the measurements are not part of the design process. The design is frozen to allow for iterating through the load measurements throughout the model in one given state. Colonia Guell was, if one ignores a small earlier hanging model, Gaudi's only design developed with the aid of a hanging model. The model was produced between 1898 and 1908 by a highly qualified team. (Tomlow, et al 1989)

The digital version, in contrast, allows simultaneous measurement and creating/editing of geometry. These measurements can directly drive other dimensions in the model. In the digital model, editing and creating the string weight is less limited by the availability and preparation of the physical material, which in the case of a very complex model can substantially slow down the process. Furthermore, the use of generative techniques allows for the rapid placement of complex string constructs and observing their behavior before investing time into an elaborate physical model.

Finally, it is possible to create the topology of the model in a frozen state to establish the pure connectivity of the model and to then subject the model to simulated gravity and

observe the form that emerges. This is a substantial advantage over the physical model, where gravity cannot be suppressed, which makes topological layouts harder to manage.

3 Precedent for geometric constraint based construction systems

3.1 Heinz Isler - Physical models for form finding measurements

Since the 1950s, Swiss engineer Heinz Isler has pioneered shell structures with minimal thickness. The lines of thrust lie exactly within the thin shell cross-section. Isler extracts the geometry in a very tedious translation process from the physical scale model and scales it to full size. Isler himself on a number of occasions came across what he thought were mistakes in the models. Isler has commented that it was a sort of not-correctness in his ideas at first, a mistake. He was unhappy that this experiment did not succeed but finally he realized that it was giving him the solution for three problems that he had not thought of. (Chilton 2000) Once the measuring is completed, it is a very time consuming process to make any alterations to the design. But physical models built in the real world also ensure that the model is not the artifact of a selective simulation strategy that may or may not include key parameters necessary to model the structure accurately. Despite the challenges they pose in achieving accuracy and scaling of material and mass, physical models do ensure a more holistic simulation of the problem than an image based computer simulation.

3.2 Antonio Gaudi - The hanging model as a method of design

Hanging models enable the designer to determine the optimal form of structures carrying loads purely in compression, particularly those consisting mainly of vaults. (Tomlow et al 1989) Although Gaudi's hanging models are the best known examples, previous, less sophisticated attempts at using hanging models were made by Heinrich Huebsch (1795-1863) and Giovanni Poleni. (Tomlow et al 1989)

The translation of the force diagram into form for some of Gaudi's projects like Sagrada Familia (not derived with a hanging model but a plaster model) was guided by two constraints: The extrusion process of the plaster model building, and the technique of a stonemason to build surfaces with straight lines of ruling. Therefore, Gaudi's Sagrada Familia combines two design constraint models coming from different ends. One is the overall structural geometry designed to be in equilibrium and in compression only, the other is the mode of construction using ruled surfaces only. There was a strong parallel between the work of the plaster mould-makers and the actual full-size construction of the window. In both cases, the straight lines in the ruled surfaces were used in a similar way to generate moulds. (Burry 2001) However, the hanging models do not implement the ruled surface constraints for the envelopes, and neither does the ruled surfaces rule take the distribution of masses along the structure into account. It requires an expert's interpretation to make them work together. Gaudi saw forms and once he had determined them mentally, he sought the means to transform them into physical, buildable objects.

(Giralt-Miracle 2002) A combination of multiple constraint models in a digital simulation would yield a wider range of exploration in both overall proportion and composition in correspondence with the surface of the building components.

3.3 Frank O. Gehry - Developable surface strips for approximating freeform surfaces

Developable strips for approximating freeform surfaces is another geometric constraint strategy allowing the exploration of free form while providing a building strategy. Developable surfaces are thus an important element of the arsenal of techniques used by Frank O. Gehry Associates since the early 1990s for constructability modeling. (Shelden 2002) This technique is not an example of a form finding process, but rather of using geometric properties constraints. These could be linked to a form finding principle to produce structurally efficient shapes that could be built economically with sheet based materials. In Gehry's practice, however, the form driver is not structural optimization, but sculptural expressiveness.

3.4 Joerg Schlaich, Hans Schober - Quadrangular planar panel solutions for freeform surfaces

The use of translation surfaces allows the fabrication of double curved surfaces with quadrangular, planar elements and constitutes a further geometric constraint modeling principle. It has been perfected by the office of Schlaich and Bergemann since the early 1990s. A robust fabrication constraint strategy is a necessity to allow for the geometric adjustments to achieve buckling resistance in the very thin glass shell domes the firm builds. This procedure allows for the use of planar glazing and plane formwork elements, a major prerequisite for the economic realization of glazed, opaque or concrete spatial structures. (Schober 2002)

4 Catenary simulation

4.1 Equation based catenaries

It is possible to calculate a catenary curve between two support points given a string length based on a parametric equation. (Figure 2) The hyperbolic cosine,

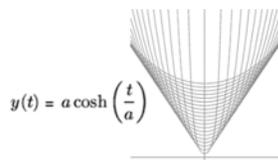


Figure 2. Catenary equation and family of catenary curve plots (Weisstein 1999)

forms the basis for the catenary equation and allows the calculation of any point on a hanging string of uniform weight, which is supported at two points. Where $t = 0$

corresponds to the vertex, and a is a parameter that determines how quickly the catenary "opens up." (Weisstein 1999) The amount of "sagging" is related to the string length and the distance between the supports. The equation can be adapted to uneven support points as well. Paul Cella noticed that textbooks of mathematics, mechanics and engineering practice produced what appeared to be a settled conclusion: When the supports of a catenary are at different elevations the mathematical complexity precludes a theoretically correct solution and a parabolic approximation is the recommended approach. (Cella 1999) He subsequently derived the catenary equations to calculate the uneven support catenary problem. (Figure 3)

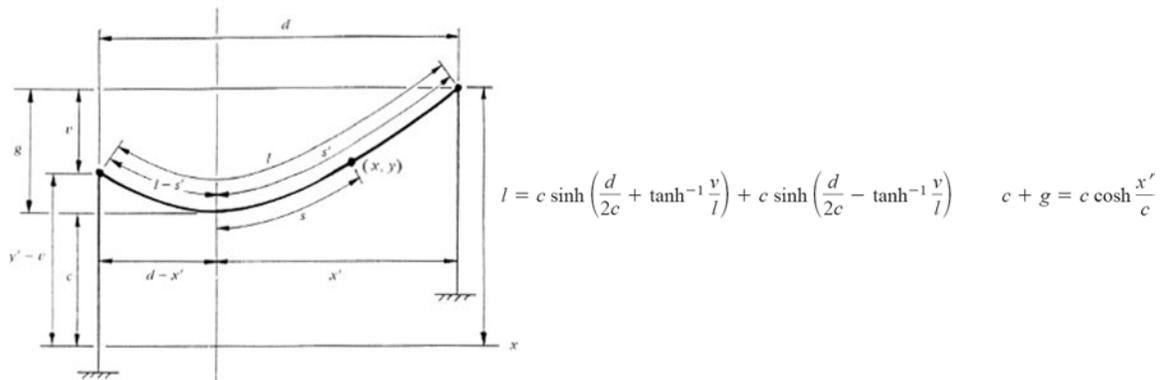


Figure 3. Calculation of a catenary curve between uneven supports (Cella 1999)

What the catenary equation based approach does not provide is a way of solving undetermined structures, for instance if four strings are joined in one node and no single solution exists. The subparts of the catenary between support points can still be solved equation-based, but the location of the supports in the general case can only found through a solver-based approach. This is where the use of a solver is necessary in order to determine the overall geometry for the equilibrium of forces in the structure.

4.2 The computational model used for simulating hanging models

A more general approach to the problem is using a particle-based approach that represents a punctual mass in space and has a position and velocity as well as an acceleration property. Based on Newton's law, a force acting on a body causes acceleration, which is inversely proportional to the mass of the body in the direction the force is applied. We can formulate a system of equations that can be integrated analytically to solve for the position of a point mass with respect to time. With the introduction of gravity, linear spring force and viscous drag it is possible to construct a particle spring system with simulated gravity. For the simple determined cases, analytical integration works. But in more complex cases, it is necessary to integrate numerically. Different methods have been developed, among them Euler method, midpoint, Runge Kutta. (Baraff and Witkin 1999)

The author used an explicit Euler solver for mesh simulation, which is satisfactory for

relatively low stiffness of the springs. During a workshop at MIT in the Spring of 2004 conducted by John Ochsendorf, Axel Kilian, Barbara Cutler, Eric and Marty Demaine, a second implementation was written which can handle meshes with very high stiffness and uses an implicit version of Euler and Runge Kutta to get stable solutions. Cloth strongly resists stretching motions while being comparatively permissive in allowing bending or shearing motions. This results in a “stiff” underlying differential equation of motion. (Press et al 1986) Explicit methods are ill-suited to solving stiff equations because they require many small steps to stably advance the simulation forward in time. (Baraff and Witkin 1998) Strings in the hanging model have similar characteristics as cloth as they have very stiff, meaning non-stretching segments making up the string.

This computational method allows the interactive construction of string like constructs out of particles and spring elements that approximate physical hanging model behavior when subjected to gravity. This approach is not new but well established in the computer graphics community and the animation industry. Novel is to use it as an interactive modeling environment for designers that allows not only optimization but also playful exploration of the evolving structure. The system is being referred to either as mass spring system or particle system. For the remainder of the paper the term Particle Spring system will be used.

5 Translation from an abstract string model to a volumetric envelope

Architects who have used hanging models struggled with the translation of the two-dimensional elements of the network of strings into the complex spatial curvature of the three-dimensional surfaces necessary to stay true to the model. In “Das Model” it is stated, “Apparently as a side effect of this struggle to follow constructional forms based on hanging models, for the first time in architectural history the hyperbolic paraboloid form was tried out in a building.” (Tomlow et al 1989)

The hanging model provides a line model for the load paths for a given distribution of weight. However, it does not specify where the envelope lies in correspondence to the load path.

In general, the self-weight of the load-bearing member contributes only negligible amounts to the structure locally and therefore does not substantially affect the hanging curve form. If there is no load present other than the weight of the structure itself, the self-weight becomes the dominant form giving factor. The cross section has to provide enough area for the forces traveling through it. A further optimization of the structure, for example with the aim of achieving uniform compressional loading throughout the same material, which would be possible by varying thickness, was not undertaken by Gaudi. (Tomlow et al 1989)

The digital hanging model presented here does create varying thickness of the extrusion along the members based on the forces present in each member. The resolution of the simulation is also a factor in determining the volumetric form. In order to keep the number of particles in the simulation low, the number of particles a hanging line is

usually relatively low. (Figure 4) This leads to polyline forms with straight spring sections in between. If the resolution is too low, it significantly offsets the load path from the ideal catenary. In a later version, dynamic subdivision of the springs based on offset measurements from the ideal catenary will be implemented.

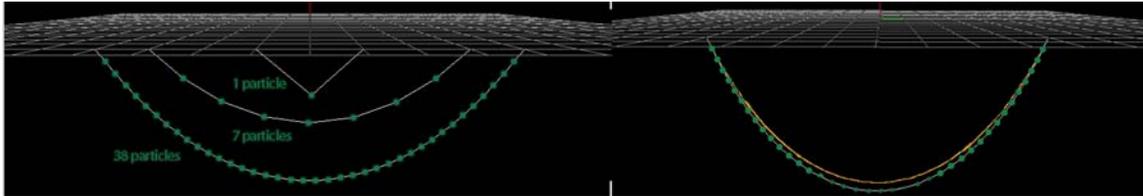


Figure 4. Hanging chains with varying subdivisions of particles (left). The hanging chain approximates the catenary. Some stretch occurs due to non-stiff springs due to the explicit solver (right).

5.1 Linear extrusion

The most straightforward translation from the line skeleton into a volumetric entity is an extrusion of a profile along the spring vectors with its diameter being approximately scaled to the forces present in the particular section of the string. Additional factors for determining the cross section are local and global buckling. (Figure 5)

The shape of the extrusion can be varied to produce different cross sections depending on what material is being used. The load path must lie completely within the geometric envelope in order to keep the structure in equilibrium. Furthermore, if the material cross section is not supposed to be subject to tension, the load path has to lie within the innermost third of a symmetrical cross section.

A bigger question is the development of the intersection between members of differing cross sections and spatial orientations. At acute angles, the intersections can become quite large in comparison to the members themselves and create difficulties in applying a uniform joint system. Parametric studies have been done to study joint systems to take this variation into account. (Figure 6)

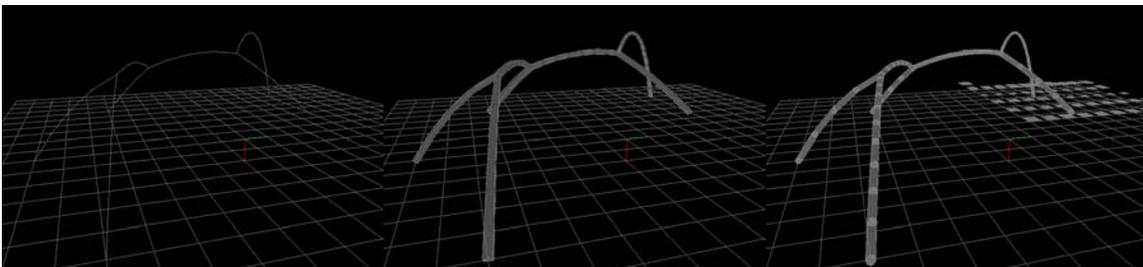


Figure 5. Different linear translation strategies programmatically linked to the form finding path: (a) path only, (b) square extrusion scaled to allowable stresses, (c) round section and ball points and unfolded cylinder surfaces



Figure 6. Joint studies from a previous project for intersections of members of unequal diameter in different spatial configurations using a parametric representation

5.2 Grid mesh strategies

As soon as the springs form a network that is more than a linear linked chain of springs and particles the structural behavior becomes more complex. No longer is there one unique load path. Multiple paths between support and loads are possible and no single determined solution exists. Also, the behavior approximates that of a shell with increasing mesh density. An additional artifact of the simulation approach becomes apparent, as the spring approach does not ensure tension only members. If the distance between two particles falls below the rest length of the spring the spring generates compression forces. Detecting the state of the spring at any point and alerting the viewer of its compression state or alternatively removing it from the solution can avoid this problem.

In determining the thickness of shells that are approximated by grid meshes one can refer to Heinz Isler's work. Isler identifies instabilities in his shells as follows: First, at the supports second, due to general buckling; third, due to local buckling of the free edge (for which the counter curvature is so important); and finally, due to other modes. (Chilton 2000) Isler gives his general equation for shell buckling as

$$P_k = c E (t/r)^x \geq s P_{eff}$$

(P_k critical buckling load, c a modification factor, E modulus of elasticity, t shell thickness, r local radius of curvature, x power of curvature, s safety factor) (Chilton 2000)

The author explored a less rigorous, visual approach to testing the stability of a simulated form. By adding additional forces besides the vertical force of gravity, the structures can be triggered to sway. (Figure 8) The relative amount of swaying of each particle is traced for the previous 200 positions. This allows a visual comparison of the displacement of the

structure in different areas. Stable structures will show little, uniform displacements. Instability becomes apparent when traces vary widely. This approach could be developed further to integrate proper wind load and earthquake displacements. For now it is an interactive, non complex way to explore a form for structural redundancy beyond one optimal load case.

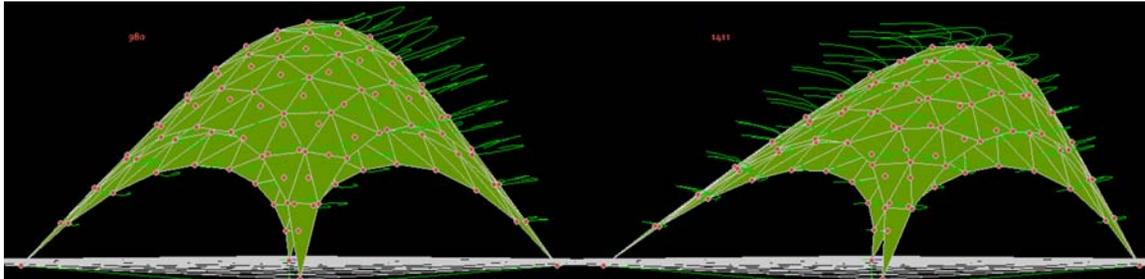


Figure 7. Simple instability test through varying gravity and tracing the position of each particle through a history trace

5.3 Integration of form finding and construction constraints into design

As illustrated in the precedent examples of Isler, Gehry, Schlaich and Gaudi, the integration of construction constraint approaches is generally solved in the following way: The form finding process defines the overall geometry, i.e. the skeleton of the structure, whereas the constraint geometries ride on top of the primary geometry, solving their constraints with regards to the overall geometry. Ideally, this combination results in constructible and structurally feasible geometries that allow the designer to focus on the expression of the overall design. In reality, the material properties rarely factor into the form finding process in design.

Some recent work in the area of tensile structures has addressed the integration of material properties into form finding and pattern generation. One method determines cutting patterns of membrane structures by taking into account the materials' viscoelasticity. (Fujiwara, Ohsaki, Uetani 2001)

The question is how much designer input is required at the early stages of design. While the designer should not be over-constrained in her or his freedom, the designer should not be overwhelmed with too specific input requests when still in the explorative stage of the design process.

6 Examples

6.1 Validation examples of the hanging model

In order to validate the results of the digital hanging model, physical models of the digitally derived geometry were built and subjected to proportional the same load distribution as simulated in the digital model. A comparison of the geometries proved the

model to be satisfactorily close to the model constructed on the basis of the digital model.

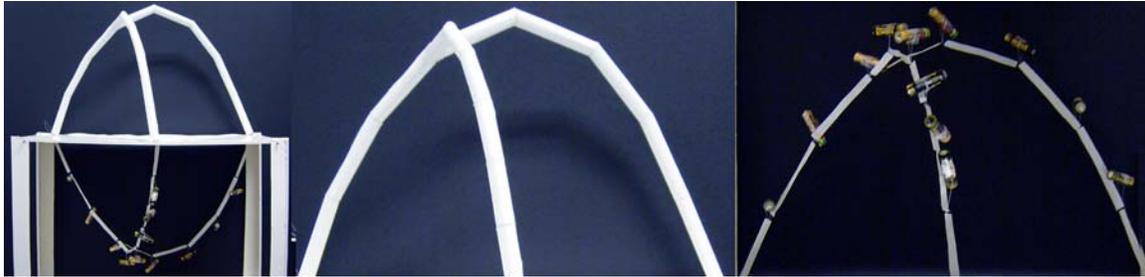


Figure 8. Physical hanging model test. (left) Paper tubes constructed on the basis of the digital model (right) The physical hanging model based on the same dimensions.

6.2 Roof example

An example model was produced from a digital hanging model and fabricated to validate the approach. For the translation of the wire frame model into a surface, rhino was used as an intermediary.

The physical model was subjected to vertical loads similar to the forces in the form finding simulation and the response of the supports was observed. The deflection due to the accumulated loads of the shell clearly shows at the two middle supports of the roof in the physical model, which corresponds to the force distribution in the form finding model. In addition, the physical model clearly demonstrates the structure's vulnerability to forces other than gravity. When the structure is loaded laterally, it is much more elastic in its response than under a vertical load and does not act as a shell.

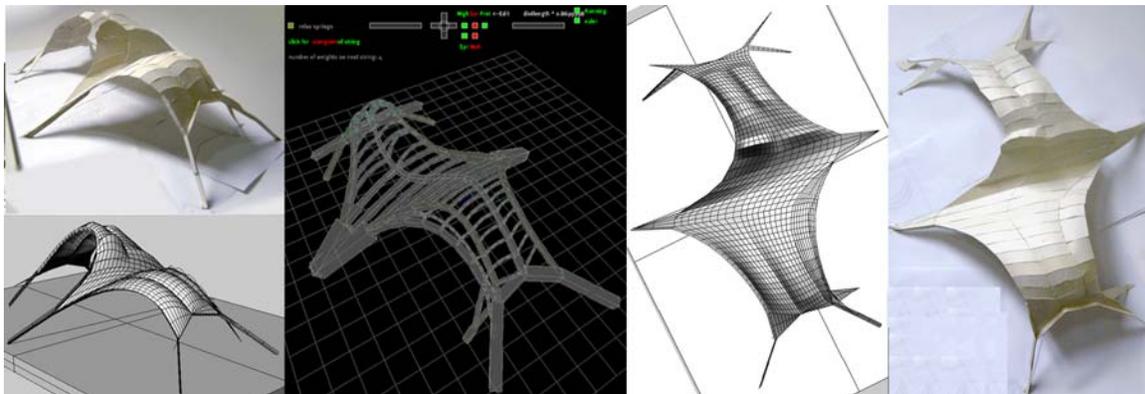


Figure 9. Roof example design using a wireframe based form finding approach with subsequent lofting and flattening for fabrication of the physical panels

7 Designing in dynamics vs. analytical approach. Design by discovery

Current design software supports the creation of geometry through geometric operations

load path finding, material distribution and testing for structural redundancy will give a much wider range of possible outcomes and allows for designer choice in the optimization.

8.1 Load path

The load paths in a structure are very much dependent on the topology of the mesh and the geometry of the individual members. Often a topology change to avoid a singular dominant load path does much more to make a form more efficient than the geometric optimization of the starting topology. For instance Isler was surprised to find that 90% of the loads in his shells are traveling into the supports in the corners of his shells. The edge supports get only a fraction of the weight. This has to do with the different stiffness of the shell areas and subsequent variant resistance to loading. Therefore optimizing compression-only structures does not guarantee evenly distributed loads. In order to achieve uniformly distributed loads in a structure the thickness has to be adapted in accordance to the loads present after the initial form finding.

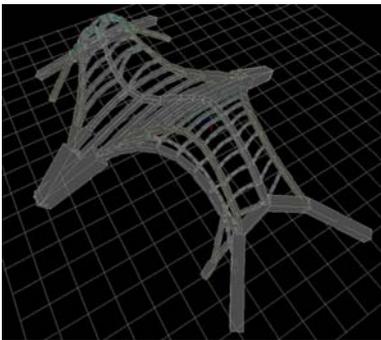


Figure 11. Load paths emerge through form finding (load proportional to the area of the cross section)

8.2 Mesh topology

The mesh topology has a substantial effect on the form of the structure and on the distribution of forces within it. The mesh topology fundamentally influences the performance of the structure. To optimize a structure cannot only mean to find the most efficient form for a given topology, but to find the most optimal topology for a given load case. This introduces the notion of topology finding or structure generation in addition to form finding.

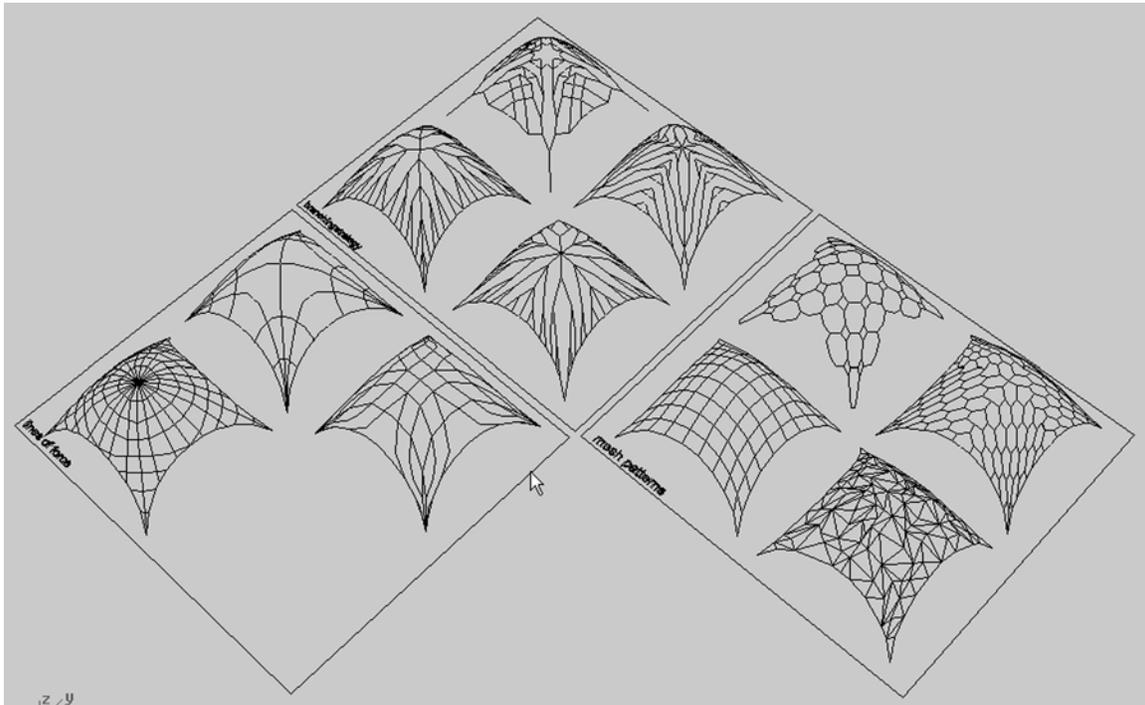


Figure 12. Variations on possible meshes for a dome with a square footprint

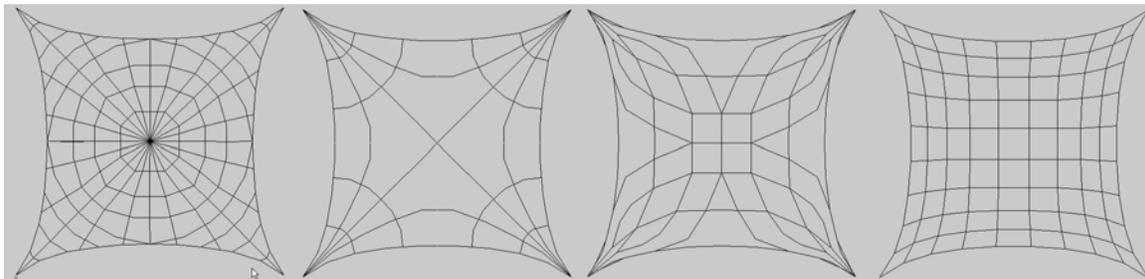


Figure 13. Details of different mesh patterns for a square footprint

8.3 Volumetric distribution

Adding material in response to the forces that need to be handled may affect the load distribution if the required cross section does not correspond with the load initially placed for the member. In most cases, the member weight will be insignificant compared to the load traveling through it. But there might be cases where another iteration of adjusting the form to the new load scenario is necessary. Minimum dimensions necessary for construction might also limit the ability to exactly match the load distribution in the abstract model.

8.4 Solid extrusions vs. hollow structure

The load path does not have to be embedded in material as long as the center of gravity of

the cross section corresponds approximately to the location of the load path. This was already realized in domes in the Renaissance and used for double shell construction. This principle is well established in space frame and truss design where the tension member and the compression member are separated by a void that is connected through diagonals.

9 Optimization as a part of design

The notion of optimization as a design support tool is questionable if it remains unchecked. Initial design moves often are exploratory in nature and may or may not have an impact on the eventual outcome. To optimize an intermediary design move may cause paths to be abandoned or curtailed prematurely. It maybe more appropriate to speak of discovery rather than optimization. Design discovery could be defined as opening up potential design paths to the designer in the light of environmental influences acting on the design such as for instance gravity. The effect of such external parameters on the design may be mediated or weighted accordingly.

Optimization should not be the sole driver in design, as choosing an optimization objective is already part of the design choice. To choose a goal is to set a design process. Although it might seem like structural performance and material usage are out of the question for design considerations, some of the most radical differences in design approach constitute themselves in the differing positions on structural rational and material efficiency. What often is missing in design optimization is the variation of the starting premise.

Kristina Shea's Eiform is one of the few examples where the optimization is coupled to a generative technique, allowing the recreation of the structure topology based on the performance analysis. (Shea and Cagan 1997) Currently the form finding tool in development based on the workshop does some dynamic editing of the mesh topology based on tension/compression distribution in the structure. If a strut goes into compression it is eliminated. Over the course of the simulation only struts in tension remain.

More research into the dynamic topology response is needed along the lines of Kristi Shea's research in order to make optimization a true component of design exploration and to allow for direct intervention by the designer.

10 The results of the workshop

In the spring semester of 2004, instructors and students from civil engineering, architecture, computer science and computer graphics conducted a workshop to investigate the problem from an interdisciplinary perspective.

A robust and scalable implicit solver was implemented in C++. It allows the handling of larger particle spring meshes and faster processing times. The application was tested in its initial stages and will be applied in a design context in the near future. For validation purposes a spherical roof shell structure was modeled and evaluated in the software.

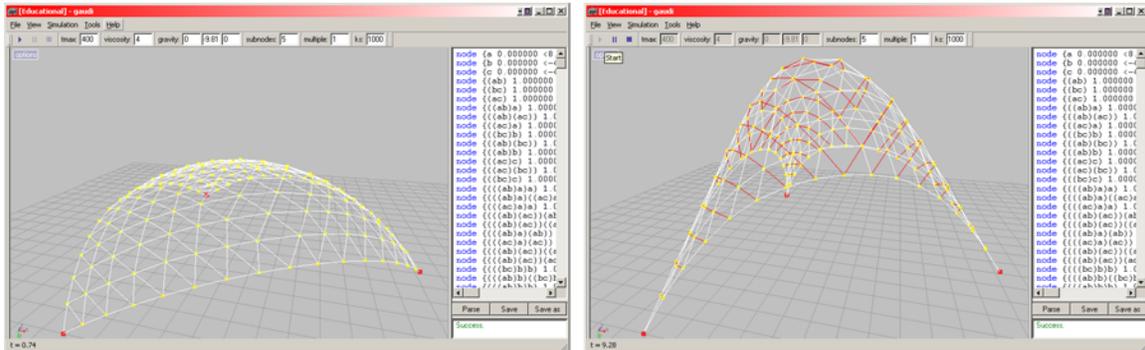


Figure 14. C++ implementation of an implicit solver, the outcome of the workshop, showing a spherical dome before and after releasing it to gravity

11 Conclusion

The paper lays out the potential for the integration of form finding techniques with fabrication strategies in a digital, integrated, expandable environment.

Form finding techniques in an interactive digital modeling environment can support the design process by giving continuous feedback to the designer, allowing the designer to integrate structural principles into the creation of form rather than to structurally optimize the finished form at the end of the design process. Digital simulation makes a range of numerical outputs of the form available for the generation of additional geometric information (like a building envelope) in response to the forces present. However, the approach to design should not a priori be driven by form finding or structural optimization. The design tools should always allow the designer to intervene and define design optimization principles.

It is certainly not the goal of the approach outlined in this paper to promote a certain style or to create a “Gaudi design machine”. The goal is to understand that expressing design goals as computational principles can support interactive design exploration and enhance the design experience. If the tools can provide a higher level of sophistication in doing so and enable handling complex competing design constraints in an interactive way, they will stimulate and challenge the notion of the design process in modeling environments today.

12 Future work

The tool will be further developed and tested in a design studio and structures class. A larger research goal is the reversal of the design direction. For instance can fabrication issues or patterns drive the form finding process? Gaudi’s compression only structural geometry might be competing with ruled surface construction constraints. A way to integrate both would be ideal - a balance between the structural efficiency, the construction constraints and the distribution of mass in correspondence to the overall structure would be ideal.

13 Acknowledgments

I would like to thank the initiator of the Spring 2004 MIT workshop Prof. John Ochsendorf and my co-instructors Dr. Barbara Cutler, Prof. Eric Demaine, Marty Demaine and Simon Greenwold as well as the workshop participants who collaborated to create the C++ application. The students were Ryo Shimizo, Eric Tung, Emily Vincent, Philippe Block, Amanda Parkes, Kyle Steinfeld, Sotirios Kotsopoulos. Participants and instructors of the interdisciplinary workshop were from architecture, computer science, civil engineering and computer graphics. An earlier 2D version of the tool was developed by the author together with Dan Chak and Megan Galbraith in the fall of 2002. The java programming for the prototype shown here was done in processing, a 3D programming environment developed for the design community by Benjamin Fry and Casey Reas at the MIT Media Lab, Cambridge, USA. The workshop will be continued in the fall of 2004.

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